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Conference Paper in AIP Conference Proceedings · October 2019

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Cite as: AIP Conference Proceedings **2159**, 030017 (2019); <https://doi.org/10.1063/1.5127482>
Published Online: 02 October 2019

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Exact controllability of semilinear control systems involving conformable fractional derivatives

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Abstract. In this paper, we discuss the exact controllability of conformable fractional semilinear control systems. First, the existence and uniqueness of the solution for such control systems are proved. Second, we establish the exact controllability results for the conformable fractional semilinear control system using fixed point theorem, assuming that the associated linear system is exactly controllable. Finally, a typical example is provided to show the effectiveness of the obtained results.

INTRODUCTION

In different fields including Engineering, control theory and applied mathematics, controllability ideas have played crucial roles. It was in 1960 when Kalman [3] first defined controllability as moving systems from basic state value to any value at late time. In order to better deal with control systems controllability meaning was divided into 2 concepts: an exact and approximate one. The aim behind these concepts is the presence of control systems which are approximately controllable but are not complete (see Fattorini [4]). Thus, the exact controllability is the same as it was defined by Kalman. However the approximate controllability is defined as shifting control systems into small neighborhood of any point at terminal time. Later on, several scholars have conducted outstanding studies in the sake of obtaining convenient controllability conditions control systems (linear and nonlinear), see for instance [5, 6, 7, 8] and the references cited therein.

Using fractional differential equations we can modelize many problems in the real world for more details refer to [9, 10, 11]. This new calculus has attracted the mathematicians to concentrate on showing better results. Various researchers have worked on extending The concept of controllability into fractional control systems. For example, Sakthivel et al. [12] prove the controllability of nonlinear fractional systems by using utilized fixed point theorem. Mahmudov [13] derives the conditions of Approximate controllability for fractional Sobolev-type evolution equations in Banach spaces. Jneid [14] derived sufficient conditions of approximate controllability for semilinear integro-differential systems of fractional order with nonlocal conditions by using compact semigroup operator and Schauder fixed point theorem.

The most preceding efforts concerning controllability problems for fractional systems have been used Riemann liouville and Caputo derivatives. However, regarding the conformable fractional derivative [15], there was no study done on it. Finally, the purpose of this work is to bring the attention of researchers on these systems..

Formulating of the problem

In this paper, we study the following semi-linear control system involving conformable fractional derivative

$$\begin{cases} D^q z_t = Az_t + Bu_t + f(t, z_t, u_t), & 0 < t \leq T, \quad 0 < q \leq 1, \\ z(0) = z_0, \end{cases} \quad (1)$$

where z and u are state and control processes and $D^q z_t$ represents the conformable fractional derivative of z at t . Let following assumptions be hold.

- (A1) Z and U are two Hilbert spaces.
(A2) A is the infinitesimal generator of a C_0 -semigroup $\Phi(t)$, $t \geq 0$.
(A3) B is a bounded linear operator from U to X ;
(A4) The mapping $f : [0, T] \times Z \times U \rightarrow Z$ is continuous and
(i) there is $M > 0$ such that $\|f(t, z, u)\| \leq M \forall (t, z, u) \in [0, T] \times Z \times U$;
(ii) there is $N > 0$ so that

$$\|f(t, y, u) - f(t, z, v)\| \leq K(\|y - z\| + \|u - v\|)$$

for all $t \in [0, T]$, $u, v \in U$ and $y, z \in Z$,

- (A5) $U_{\text{ad}} = C(0, T; U)$ is the space of all continuous functions from $[0, T]$ to U .

Define the controllability operator in conformable sense W_r^t as

$$W_r^t = \int_r^t s^{q-1} \left[\Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) B \right]^* ds, \quad 0 \leq r \leq t \leq T. \quad (2)$$

where $*$ indicates the adjoint.

- (A6) W_0^T is coercive, i.e., there exists a $\mu > 0$ such that $\langle W_0^T z, z \rangle \geq \mu \|z\|^2$, $\forall z \in Z$, and consequently $(W_0^T)^{-1}$ exists and $\|(W_0^T)^{-1}\| \leq \frac{1}{\mu}$.

The above assumptions show that the equation (1) has a unique mild solution for every $u \in U_{\text{ad}}$ and $z_0 \in Z$ (see, Jaiswal and Bahuguna [1]), that is, there is a unique continuous function z from $[0, T]$ to Z such that

$$z_t = \Phi\left(\frac{t^q}{q}\right) z_0 + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) (Bu_s + f(s, z_s, u_s)) ds. \quad (3)$$

Let

$$R(z_0, T) = \{h \in Z : \exists u \in U_{\text{ad}} \text{ such that } h = z_T\}. \quad (4)$$

Owing to Bashirov and Jneid [5], the control system (1) is said to be exact controllable on U_{ad} if $R(z_0, T) = Z$ for all $z_0 \in Z$. In a similar way, the control system (1) is said to be approximate controllable on U_{ad} if $\overline{R(z_0, T)} = Z$ for all $z_0 \in Z$, where $\overline{R(z_0, T)}$ represents the closure of $R(z_0, T)$. In this paper, we focus on the concept of exact controllability and we find that under the assumptions (A1)–(A6) and an additional condition which will be given in the following section, the semi-linear control system (1) is exact controllable.

Main Result

The product space $C(0, T; Z) \times U_{\text{ad}}$ is a Banach space equipped with the well known supremum norm in operator

$$\|(\cdot, \cdot)\|_{C(0, T; Z) \times U_{\text{ad}}} = \|\cdot\|_{C(0, T; Z)} + \|\cdot\|_{U_{\text{ad}}}.$$

Lemma 1. Under the assumptions (A1), (A2) and (A3), the following holds

$$\|W_0^t\| \leq \|W_0^T\| \quad \text{and} \quad 0 \leq t \leq T.$$

Proof: It is not hard to show that $W_0^t = (W_0^t)^*$ and for all $z \in Z$, $\langle W_0^t z, z \rangle \geq 0$. Then,

$$\|W_0^t\| = \sup_{\|z\| \leq 1} |\langle W_0^t z, z \rangle|.$$

Therefore

$$\begin{aligned}
\langle W_0^T z, z \rangle &= \left\langle \int_0^T s^{q-1} \left[\Phi\left(\frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* ds z, z \right\rangle \\
&= \left\langle \int_0^t s^{q-1} \left[\Phi\left(\frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* ds z, z \right\rangle + \left\langle \int_t^T s^{q-1} \left[\Phi\left(\frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* ds z, z \right\rangle \\
&= \langle W_0^t z, z \rangle + \left\langle \int_t^T s^{q-1} \left[\Phi\left(\frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* ds z, z \right\rangle,
\end{aligned}$$

where,

$$\begin{aligned}
\left\langle \int_t^T s^{q-1} \left[\Phi\left(\frac{s^q}{q}\right) B \right] \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* ds z, z \right\rangle &= \int_t^T s^{q-1} \langle \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* z, \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* z \rangle ds \\
&= \int_t^T s^{q-1} \| \left[\Phi\left(\frac{s^q}{q}\right) B \right]^* z \|^2 ds \geq 0.
\end{aligned}$$

Hence, $\langle W_0^t z, z \rangle \leq \langle W_0^T z, z \rangle$, yields $\|W_0^t\| \leq \|W_0^T\|$.

□ The proof of the following lemma is a minor modification of similar lemmas appear in several papers, for instance, Mahmudov [2] and Bashirov and Jneid [5], etc..

Lemma 2. Under the assumption (A1)–(A6), let $h \in Z$. Then for the nonlinear operator $F : C(0, T; Z) \times U_{\text{ad}} \rightarrow C(0, T; Z) \times U_{\text{ad}}$ given as

$$F(y, v)(t) = (Y(t), V(t)), \quad 0 \leq t \leq T,$$

where

$$\begin{aligned}
Y(t) &= -W_0^t \Phi^* \left(\frac{T^q}{q} - \frac{t^q}{q} \right) [W_0^T]^{-1} \int_0^T s^{q-1} \Phi \left(\frac{T^q}{q} - \frac{s^q}{q} \right) f(s, y_s, v_s) ds \\
&\quad + \int_0^t s^{q-1} \Phi \left(\frac{t^q}{q} - \frac{s^q}{q} \right) f(s, y_s, v_s) ds
\end{aligned} \tag{5}$$

$$V(t) = B^* \Phi^* \left(\frac{T^q}{q} - \frac{t^q}{q} \right) [W_0^T]^{-1} \left(h - \Phi^* \left(\frac{T^q}{q} \right) z_0 - \int_0^T s^{q-1} \Phi \left(\frac{T^q}{q} - \frac{s^q}{q} \right) f(s, y_s, v_s) ds \right), \tag{6}$$

the following estimation holds:

$$\|F(y, v) - F(z, w)\| \leq \frac{T^q}{q} \left(1 + \frac{\|W_0^T\|L}{\mu} + \frac{\|B\|L}{\mu} \right) LK (\|y - z\| + \|v - w\|),$$

where

$$L = \sup_{0 \leq t \leq T} \left\| \Phi \left(\frac{t^q}{q} \right) \right\|.$$

Proof: Let $(y, v), (z, w) \in (0, T; Z) \times U_{\text{ad}}$ with $F(y, v) = (Y, V)$ and $F(z, w) = (Z, W)$. Then,

$$\|F(y, v) - F(z, w)\|_{C(0, T; Z) \times U_{\text{ad}}} = \|Y - Z\|_{C(0, T; Z)} + \|V - W\|_{U_{\text{ad}}}$$

Here, $\|Y - Z\|_{C(0,T;Z)}$ can be estimated as follows:

$$\begin{aligned}
\|Y - Z\| &= \sup_{t \in [0,T]} \left\| W_0^t \Phi^* \left(\frac{T^q}{q} - \frac{t^q}{q} \right) [W_0^T]^{-1} \int_0^T s^{q-1} \Phi \left(\frac{T^q}{q} - \frac{s^q}{q} \right) (f(s, y_s, v_s) - f(s, z_s, w_s)) ds \right. \\
&\quad \left. + \int_0^t s^{q-1} \Phi \left(\frac{t^q}{q} - \frac{s^q}{q} \right) (f(s, y_s, v_s) - f(s, z_s, w_s)) ds \right\| \\
&\leq \sup_{t \in [0,T]} \left(L + L^2 \|W_0^t\| \cdot \|[W_0^T]^{-1}\| \right) \int_0^T s^{q-1} \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\leq \left(1 + \frac{\|W_0^T\| L}{\mu} \right) LK \int_0^T s^{q-1} (\|y_s - z_s\| + \|v_s - w_s\|) ds \\
&\leq \frac{T^q}{q} \left(1 + \frac{\|W_0^T\| L}{\mu} \right) LK (\|y - z\| + \|v - w\|).
\end{aligned} \tag{7}$$

In like manner, for $\|V - W\|_{U_{ad}}$ we have

$$\begin{aligned}
\|V - W\| &= \sup_{t \in [0,T]} \left\| B^* \Phi^* \left(\frac{T^q}{q} - \frac{t^q}{q} \right) [W_0^T]^{-1} \int_0^T s^{q-1} \Phi \left(\frac{T^q}{q} - \frac{s^q}{q} \right) (f(s, y_s, v_s) - f(s, z_s, w_s)) ds \right\| \\
&\leq \sup_{t \in [0,T]} \left(L^2 \|B\| \cdot \|[W_0^T]^{-1}\| \right) \int_0^T s^{q-1} \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\leq \frac{1}{\mu} L^2 \|B\| K \int_0^T s^{q-1} (\|y_s - z_s\| + \|v_s - w_s\|) ds \\
&\leq \frac{T^q}{q} \frac{\|B\| L}{\gamma} LK (\|y - z\| + \|v - w\|).
\end{aligned} \tag{8}$$

Combining (5) and (6) we get the desired inequality. $\square$

Lemma 3. Under the assumptions (A1)–(A6), if

$$\frac{T^q}{q} \left(1 + \frac{\|W_0^T\| L}{\mu} + \frac{\|B\| L}{\mu} \right) LK < 1, \tag{9}$$

then the operator F maps $C(0, T; Z) \times U_{ad}$ into $C(0, T; Z) \times U_{ad}$ and has a unique fixed point $(z, u) \in C(0, T; Z) \times U_{ad}$.

Proof: By Lemma 2, G is a contraction mapping. Hence, it has a fixed point. $\square$

Theorem 1. Under the assumptions (A1)–(A6) and (9), the semilinear system (1) is exactly controllable on $[0, T]$.

Proof: Take any $h \in Z$. Show that there is $u \in U_{ad}$ such that $h = z_T$. To this end, consider u is defined as follows

$$u_t = B^* \Phi^* \left(\frac{T^q}{q} - \frac{t^q}{q} \right) [W_0^T]^{-1} \left(h - \Phi^* \left(\frac{T^q}{q} \right) z_0 - \int_0^T s^{q-1} \Phi \left(\frac{T^q}{q} - \frac{s^q}{q} \right) f(s, z_s, u_s) ds \right) \tag{10}$$

Substituting (10) in (3) and applying Fubini's theorem we obtain

$$\begin{aligned}
z_t &= \Phi\left(\frac{T^q}{q}\right)z_0 + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) BB^* \Phi^*\left(\frac{t^q}{q} - \frac{s^q}{q}\right) \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} (h - \Phi\left(\frac{T^q}{q}\right)z_0) ds \\
&\quad - \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) BB^* \Phi^*\left(\frac{t^q}{q} - \frac{s^q}{q}\right) \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} \times \left(\int_0^T r^{q-1} \Phi\left(\frac{T^q}{q} - \frac{r^q}{q}\right) f(r, z_r, u_r) dr \right) ds \\
&\quad + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) f(s, z_s, u_s) ds \\
&= \Phi\left(\frac{t^q}{q}\right)z_0 + W_0^t \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} (h - \Phi\left(\frac{T^q}{q}\right)z_0) + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) f(s, z_s, u_s) ds \\
&\quad - \int_0^T \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) BB^* \Phi^*\left(\frac{t^q}{q} - \frac{s^q}{q}\right) \Phi^*\left(\frac{T^q}{q} - \frac{r^q}{q}\right) [W_0^T]^{-1} \Phi\left(\frac{T^q}{q} - \frac{r^q}{q}\right) f(r, z_r, u_r) ds r^{q-1} dr \\
&= \Phi\left(\frac{t^q}{q}\right)z_0 + W_0^t \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} (h - \Phi\left(\frac{T^q}{q}\right)z_0) + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) f(s, z_s, u_s) ds \\
&\quad - \int_0^T r^{q-1} W_0^t \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} \Phi\left(\frac{T^q}{q} - \frac{r^q}{q}\right) f(r, z_r, u_r) dr \\
&= \Phi\left(\frac{t^q}{q}\right)z_0 + W_0^t \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} (h - \Phi\left(\frac{T^q}{q}\right)z_0) + \int_0^t s^{q-1} \Phi\left(\frac{t^q}{q} - \frac{s^q}{q}\right) f(s, z_s, u_s) ds \\
&\quad - W_0^t \Phi^*\left(\frac{T^q}{q} - \frac{t^q}{q}\right) [W_0^T]^{-1} \int_0^T r^{q-1} \Phi\left(\frac{T^q}{q} - \frac{r^q}{q}\right) f(r, z_r, u_r) dr
\end{aligned} \tag{11}$$

Now, setting $t = T$, we obtain

$$\begin{aligned}
z_T &= \Phi\left(\frac{T^q}{q}\right)z_0 + W_0^T [W_0^T]^{-1} (h - \Phi\left(\frac{T^q}{q}\right)z_0) + \int_0^T s^{q-1} \Phi\left(\frac{T^q}{q} - \frac{s^q}{q}\right) f(s, z_s, u_s) ds \\
&\quad - W_0^T [W_0^T]^{-1} \int_0^T r^{q-1} \Phi\left(\frac{T^q}{q} - \frac{r^q}{q}\right) f(r, z_r, u_r) dr \\
&= h.
\end{aligned}$$

Therefore, there is $u \in U_{\text{ad}}$ which leads z_0 to z_T with $z_T = h$. This means that the semilinear system (1) is exactly controllable on $[0, T]$ as desired. $\square$

Example

Example 1. Consider the following conformable fractional control system

$$D_q z_t = z_t + u_t + f(t, z_t, u_t), \quad z_0 \in \mathbb{R}, \tag{12}$$

where, $0 \leq t \leq T$, $z, u \in C(0, T; \mathbb{R})$.

It can be computed that

$$\begin{aligned}
W_0^T &= \int_0^T s^{q-1} e^{\frac{t^q}{q}} BB^* e^{\frac{t^q}{q}} ds, \\
&= \int_0^T s^{q-1} e^{\frac{2s^q}{q}} ds, \\
&= \frac{1}{2} (e^{\frac{2T^q}{q}} - 1),
\end{aligned} \tag{13}$$

which is coercive for every $0 < q \leq 1$. Hence

$$\|W_0^T\| = \mu = \frac{1}{2}(e^{\frac{2Tq}{q}} - 1).$$

Therefore, the inequality (9) becomes

$$T^q(1 + e^{\frac{Tq}{q}} + \frac{2e^{\frac{Tq}{q}}}{(e^{\frac{2Tq}{q}} - 1)})e^{\frac{Tq}{q}}K < q, \quad (14)$$

If the final time T and the Lipschitz coefficient K of the function f satisfy this inequality, then the given system (12) is exactly controllable on $[0, T]$.

Conclusion

In this work, sufficient conditions for the exact controllability of semilinear control systems involving conformable fractional derivatives are derived. These conditions are obtained with the help of fixed point theorem and the natural condition that the conformable controllability operator is coercive.

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